ON THE ASYMPTOTIC BEHAVIOR OF SPECTRAL FUNCTIONS AND RESOLVENT KERNELS OF ELLIPTIC OPERATORS

BY

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ABSTRACT

Asymptotic formulas with remainder estimates are derived for spectral functions of general elliptic operators. The estimates are based on asymptotic expansion of resolvent kernels in the complex plane.

1. **Introduction.** Let Ω be an open set in real space R^n with generic point $x = (x_1, \cdots, x_n)$. We denote by $H_m(\Omega)$, $m \geqq 0$ an integer, the subclass of functions $u \in L_2(\Omega)$ with (distribution) derivatives $D^\alpha u \in L_2(\Omega)$ for all $|\alpha| \leqq m$. Here and in the following $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad D_k = -i\frac{\partial}{\partial x_k}, \qquad i = \sqrt{-1}.$$

We denote by $H_m^{loc.}(\Omega)$ the class of functions defined on Ω and belonging locally to H_m. In $H_m(\Omega)$ we introduce the norm:

$$\| u \|_{m,\Omega} = \left[\int_\Omega \sum_{|\alpha| \leqq m} \binom{m}{\alpha} | D^\alpha u |^2 dx \right]^{1/2} \tag{1.1}$$

where dx is the Lebesgue measure and the binomial coefficients

$$\binom{m}{\alpha} = m!\,(\alpha_1! \cdots \alpha_n!\,(m - |\alpha|)!)^{-1}$$

are introduced for convenience. Under this norm $H_m(\Omega)$ is a Hilbert space.

Let $\rho(x)$ be a C^∞ positive function on Ω. We denote by $d_\rho x$ the measure $\rho(x)\,dx$ and by $(\ ,\ )_\rho$ the scalar product:

$$(f, g)_\rho = \int_\Omega f(x)\overline{g(x)}\,d_\rho x \tag{1.2}$$

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We denote by $L_{2,\rho}(\Omega)$ the Hilbert space which is the completion of $C_0^\infty(\Omega)$ (class of infinitely differentiable functions with compact support in Ω) under the norm $(f,f)^{1/2}$.

Let

$$A = A(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{1.3}$$

be a linear differential operator of order m with C^∞ coefficients in Ω. We denote its principal part by $A' = A'(x,D)$. We assume that A is a positive elliptic operator and that it is ρ-formally self-adjoint. That is we assume that

$$A'(x,\xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha > 0, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n},$$

for all real $\xi = (\xi_1, \cdots, \xi_n) \neq 0$ and $x \in \Omega$, and that $(Au, v)_\rho = (u, Av)_\rho$ for all $u, v \in C_0^\infty(\Omega)$.

We denote by $\tilde{A}$ a self-adjoint realization of A in $L_{2,\rho}(\Omega)$. That is, $\tilde{A}$ is a self-adjoint operator in the Hilbert space $L_{2,\rho}(\Omega)$ with domain of definition $\mathscr{D}_{\tilde{A}}$ such that any $u \in \mathscr{D}_{\tilde{A}}$ is a solution in the *distribution* sense of the differential equation:

$$A(x,D)u = \tilde{A}u \tag{1.4}$$

By well known regularity results for weak solutions of elliptic equations (e.g. [1]) it follows from (1.4) that $\mathscr{D}_{\tilde{A}} \subset H_m^{loc.}(\Omega)$. More generally, since $\tilde{A}^k$ is a realization of A^k:

$$\mathscr{D}_{\tilde{A}^k} \subset H_{km}^{loc.}(\Omega) \text{ for } k = 1, 2, \cdots. \tag{1.5}$$

Assume that the self-adjoint realization $\tilde{A}$ is bounded from below and let $\{E_t\}$ be its spectral resolution (normalized by left continuity):

$$\tilde{A} = \int_{-\infty}^{\infty} t dE_t .$$

It is (essentially) well known that E_t is an integral operation:

$$E_t f = \int_\Omega e(t; x, y) f(y) d_\rho y, \qquad f \in L_{2,\rho}(\Omega),$$

with a C^∞ Carleman type kernel $e(t; x, y)$ called the spectral function of $\tilde{A}$ (see Gårding [9] and section 2 of this paper). In particular $e(t; x, x)$ is a real non-negative non-decreasing function of t.

The problem of the asymptotic behavior of spectral functions was first investigated by Carleman [7] for a class of second order elliptic operators. In more recent years the problem was studied by many authors for more general operators (e.g. [14; 15] [8; 9; 10] [5, 6], [12], [4], [2]). In particular Gårding [8] proved that in the general situation discussed above, when $\rho = 1$:

$$e(t; x, x) - c(x)t^{n/m} = o(t^{n/m}) \text{ as } t \to +\infty, \tag{1.6}$$
$$c(x) = (2\pi)^{-n} \int_{A'(x,\xi)<1} d\xi.$$

Gårding has also shown that when A has constant coefficients then the remainder term $o(t^{n/m})$ in (1.6) can be replaced by the term: $O(t^{(n-1)/m})$.

For elliptic operators with variable coefficients a remainder estimate in the asymptotic formula (1.6) is known in the literature only for a class of second order operators. Avakumovic [3] proved that for the Laplace-Beltrami operator on a compact Riemannian manifold the remainder term $o(t^{n/2})$ in (1.6) can be replaced by $O(t^{(n-1)/2})$ (this result was proved explicitly only for $n = 3$). Recently S. Agmon proved (unpublished) that for elliptic operators of any order the following estimate for the remainder term in (1.6) holds:

$$e(t; x, x) - c(x)t^{n/m} = O(t^{(n-\theta)/m}), \qquad t \to +\infty, \tag{1.7}$$

where θ is any number $< \frac{1}{4}$ in the general case and θ any number $< \frac{3}{4}$ if the principal part A' has constant coefficients.

The main purpose of this paper is to improve further the last mentioned remainder estimate. We shall prove that (1.7) holds with any $\theta < \frac{1}{2}$ in the general case and any $\theta < 1$ if A' has constant coefficients (actually we shall prove a somewhat more general result, see Theorem 3.2). In this connection we mention that a short time after the derivation of our results we were informed by L. Hörmander that he has also obtained very recently the same remainder estimates for the spectral function. His method, however, seems to be different from the method employed by us. About our method we shall say here only that it uses a procedure of estimating kernels introduced in [2] to derive a fine asymptotic expansion theorem for resolvent kernels of elliptic operators. The proof of this expansion theorem (Theorem 3.1 and the more general Theorem 6.2) takes up most of this paper. Once this result is established the remainder estimates for spectral functions follow easily with the aid of a tauberian theorem due to Malliavin [13].

In another paper S. Agmon will use a modified approach to derive various extensions of the results of this paper. In particular, similar remainder estimates will be proved for the asymptotic distribution formula of eigenvalues. Remainder estimates will also be proved for spectral matrix functions corresponding to a self-adjoint realization of an elliptic system of differential operators. The self-adjoint realization will not be assumed to be semi-bounded.

In conclusion we wish to thank L. Hörmander for informing us about his results and for acquainting us with his recent, as yet unpublished, work [11] on spectral functions.

2. **Certain integral operators.** We shall have to deal with bounded linear operators T in $L_2(\Omega)$ such that range of T is contained in $H_m(\Omega)$ for some $m > 0$. By the closed graph theorem T is also bounded when considered as a linear transformation from $L_2(\Omega)$ into $H_j(\Omega)$, $0 \leqq j \leqq m$. The norm of T when considered as an operator: $L_2 \to H_j$ will be denoted by:

$$\| T \|_j = \| T \|_{j,\Omega} = \sup_{f \in L_2(\Omega)} \frac{\| Tf \|_{j,\Omega}}{\| f \|_{o,\Omega}}. \tag{2.1}$$

We state now one of the principal results of [2] (Theorem 3.1 of [2]) which will also play a basic role in this paper. In this connection recall that an open set Ω is said to possess the cone property if each point $x \in \Omega$ is the vertex of a spherical cone of a fixed height and opening contained in Ω.

THEOREM 2.1. *Let T be a bounded linear operator in $L_2(\Omega)$, Ω an open set in R^n possessing the cone property. Suppose that the range of T and that the range of its adjoint T^* are contained in $H_m(\Omega)$ for some $m > n$. Then T is an integral operator,*

$$Tf = \int_\Omega K(x,y) f(y)\, dy, \qquad f \in L_2(\Omega),$$

with a continuous and bounded kernel $K(x,y)$ satisfying

$$| K(x,y) | \leqq \gamma_0 (\| T \|_m + \| T^* \|_m)^{n/m} \| T \|_0^{1-n/m} \tag{2.2}$$

where γ_0 is a constant depending only on m, n and on the dimensions of the cone in the cone property of Ω.

Using the last theorem one can easily prove the existence of a continuous kernel in the more general situation when T is a bounded linear operator in $L_{2,\rho}(\Omega)$ such that the range of T and the range of T^* are contained in $H_m^{loc}(\Omega)$, $m > n$. To see this let $\{\Omega_j\}$, $j = 1, 2, \cdots$, be a sequence of open bounded sets possessing the cone property, $\bar{\Omega}_j \subset \Omega$, $\Omega_j \subset \Omega_{j+1}$ and $\cup_j \Omega_j = \Omega$ Let $J_j \colon L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega_j)$ be the restriction operator restricting $f \in L_{2,\rho}(\Omega)$ to Ω_j. Its adjoint $J_j^* \colon L_{2,\rho}(\Omega_j) \to L_{2,\rho}(\Omega)$ is an extension operator extending $f \in L_{2,\rho}(\Omega_j)$ as zero in $\Omega - \Omega_j$. Let: $T_j = J_j T J_j^*$. It is clear that T_j can be considered as a bounded linear operator in $L_2(\Omega_j)$ and that as such it verifies all the conditions of Theorem 2.1. Applying the theorem it follows that T_j is an integral operator with a continuous kernel $K_j(x,y)$ on $\Omega_j \times \Omega_j$. It is easy to see that $K_i(x,y) \equiv K_j(x,y)$ on $\Omega_i \times \Omega_i$ for any $i < j$. Hence the kernel $K(x,y)$ defined by $\rho(y)K(x,y) = K_j(x,y)$ in $\Omega_j \times \Omega_j$ for $j = 1, 2, \cdots$, is a well defined kernel on $\Omega \times \Omega$ such that

$$Tf = \int_\Omega K(x,y) f(y)\, d_\rho y \tag{2.3}$$

for all $f \in L_{2,\rho}(\Omega)$ with compact support in Ω. Finally (2.3) actually holds for all $f \in L_{2,\rho}(\Omega)$ since K is a Carleman kernel with respect to the measure $d_\rho y$ (indeed by Sobolev's inclusion relations: $f \to (Tf)(x)$ is a bounded linear functional on $L_{2,\rho}(\Omega)$ for each fixed x which implies that $K(x,\cdot) \in L_{2,\rho}(\Omega)$). Thus we have proved:

THEOREM 2.1. bis. *Let T be a bounded linear operator in $L_{2,\rho}(\Omega)$ such that range T and range T^* are contained in $H_m^{loc.}(\Omega)$ for some $m > \dim\Omega$. Then T is an integral operator of the form* (2.3) *with continuous Carleman kernel $K(x,y)$.*

As in the introduction we consider now a self-adjoint operator $\tilde{A}$ in $L_{2,\rho}(\Omega)$ which is a realization of a ρ-formally selfadjoint elliptic differential operator A of order m. Let $R_\lambda = (\tilde{A} - \lambda)^{-1}$ be the resolvent of $\tilde{A}$ defined for every complex λ not in the spectrum of $\tilde{A}$. Using (1.5), we have:

$$\text{range}(R_\lambda) = \text{range}\,(R_\lambda^*) = \mathscr{D}_{\tilde{A}} \subset H_m^{loc.}(\Omega).$$

Hence, if $m > n$, it follows from Theorem 2.1 bis. that R_λ is an integral operator:

$$R_\lambda f = \int_\Omega R_\lambda(x,y) f(y)\, d_\rho y,$$

with a continuous Carleman kernel $R_\lambda(x,y)$. We shall refer to $R_\lambda(x,y)$ as the resolvent kernel of $\tilde{A}$.

Next assume the $\tilde{A}$ is bounded from below but impose no restriction on m. Let $\{E_t\}$ be the spectral resolution of $\tilde{A}$. For any fixed λ not in spectrum $\tilde{A}$ and $k = 1, 2, \cdots$, we write:

$$E_t = R_\lambda^k S_{t,\lambda,k} \tag{2.4}$$

where

$$S_{t,\lambda,k} = \int_{-\infty}^{t} (s-\lambda)^k dE_s$$

is a bounded operator. As before, using (1.5):

$$\text{range}\,(R_\lambda^k) = \mathscr{D}_{\tilde{A}^k} \subset H_{km}^{loc.}(\Omega). \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\text{range}(E_t) \subset \bigcap_{j=1}^{\infty} H_j^{loc.}(\Omega).$$

Hence by Theorem 2.1 bis. E_t is an integral operator with a continuous (actually C^∞) and bounded kernel $e(t; x, y)$. This proves the existence of the spectral function of $\tilde{A}$

Suppose that $\tilde{A}$ is positive and that $m > n$. In this case both the resolvent kernel $R_\lambda(x,y)$ and the spectral function $e(t;x,y)$ exist. The following relation holds:

$$R_\lambda(x,y) = \int_0^\infty (t-\lambda)^{-1} de(t;x,y) \tag{2.6}$$

where the Stieltjes integral converges absolutely. Formula (2.6) is (essentially) well known (see [10], [4]). We note that a simple proof of (2.6) can be given with the aid of Theorem 2.1. Without giving complete details we shall sketch the proof. Choose a sequence $\{\Phi_N(t)\}$ of step functions on $t \geqq 0$, each vanishing for $t \geqq t_N$ sufficiently large and satisfying:

$$|\Phi_N(t) - (t-\lambda)^{-1}| < N^{-1}, \ |\Phi_N(t) - (t-\lambda)^{-1}| < C(1+t)^{-1} \tag{2.7}$$

for $t \geqq 0$, C a suitable constant independent of N. Put:

$$T_N = \int_0^\infty [(t-\lambda)^{-1} - \Phi_N(t)] dE_t. \tag{2.8}$$

Clearly T_N is an integral operator with a kernel

$$K_N(x,y) = R_\lambda(x,y) - \int_0^\infty \Phi_N(t)\, de(t;x,y). \tag{2.9}$$

Choose any $\Omega_0 \subset\subset \Omega$, Ω_0 possessing the cone property, and let

$$J_0 : L_{2,\rho}(\Omega) \to L_{2,\rho}(\Omega_0)$$

be the restriction operator from Ω to Ω_0. Its adjoint J_0^*: $L_{2,\rho}(\Omega_0) \to L_{2,\rho}(\Omega)$ is an extension operator. Set: $T_N^0 = J_0 T_N J_0^*$. Then T_N^0 which is a bounded operator in $L_{2,\rho}(\Omega_0)$ can also be considered as a bounded operator in $L_2(\Omega_0)$. Considered as such it satisfies the conditions of Theorem 2.1. Some simple computations (using (2.8) and (2.7)) show that

$$\begin{gathered} \| T_N^0 \|_{0,\Omega_0} = O(N^{-1}), \ \| T_N^0 \|_{m,\Omega_0} = O(1), \\ \| (T_N^0)^* \|_{m,\Omega_0} = O(1) \text{ as } N \to \infty. \end{gathered} \tag{2.10}$$

Applying the estimate (2.2) to the kernel $\rho(y)K_N(x,y)$ of T_N^0 (on $\Omega_0 \times \Omega_0$) it follows from (2.10) that

$$K_N(x,y) = O(N^{-1+n/m}) = o(1) \text{ as } N \to \infty, \tag{2.11}$$

uniformly on $\Omega_0 \times \Omega_0$. From (2.11), (2.9) and (2.7) the representation formula (2.6) follows easily. In particular the absolute convergence of (2.6) for $x = y$ follows in this way by taking $\lambda = -1$ and choosing $\Phi_N \geqq 0$, using the fact that $e(t;x,x)$ is a non-decreasing function of t. The absolute convergence of (2.6) for $x \neq y$ follows from that for $x = y$ using the following easily established estimate for the total variation of $e(t;x,y)$ on any finite interval $a \leqq t \leqq b$ (see [4]):

$$\operatorname{var} e(t; x, y) \leqq [\operatorname{var} e(t; x, x) \cdot \operatorname{var} e(t; y, y)]^{1/2}.$$

3. **The main theorems.** The key result of our paper is the following asymptotic expansion theorem for resolvent kernels of elliptic operators.

THEOREM 3.1. *Let $\tilde{A}$ be a positive self-adjoint operator in $L_{2,\rho}(\Omega)$ which is the realization of a ρ-formally self-adjoint (positive) elliptic differential operator: $A(x,D) = \sum_{|\alpha| \leqq m} a_\alpha(x) D^\alpha$ of order $m > n = \dim \Omega$. For each $x \in \Omega$ define $\theta(x)$ as follows: $\theta(x) = 1$ if*

$$\sum_{|\alpha| = m} |a_\alpha(y) - a_\alpha(x)| = O(|y - x|^p) \text{ as } y \to x \tag{3.1}$$

for all integers p. Otherwise $\theta(x) = p/(p+1)$ where $p \geqq 1$ is the largest integer for which (3.1) holds. Let $R_\lambda(x,y)$ be the resolvent kernel of $\tilde{A}$. Denote by $d(\lambda)$ the distance of λ from the positive axis ($d(\lambda) = |\lambda|$ if $\operatorname{Re}\lambda \leqq 0$, $d(\lambda) = |\operatorname{Im} \lambda|$ if $\operatorname{Re}\lambda \geqq 0$). Then $R_\lambda(x,x)$ possesses an asymptotic expansion of the form:

$$R_\lambda(x,x) \sim (-\lambda)^{n/m-1} \sum_{j=0}^{\infty} c_j(x)(-\lambda)^{-j/m} \tag{3.2}$$

valid for $\lambda \to \infty$ in the region: $|\lambda| \geqq 1$, $d(\lambda) \geqq |\lambda|^{1-(\theta(x))/m+\varepsilon}$ where ε is any given positive number, uniformly in x in every compact subset of Ω. That is, for any integer $N \geqq 1$:

$$\left|(-\lambda)^{1-n/m} R_\lambda(x,x) - \sum_{j=0}^{N-1} c_j(x)(-\lambda)^{-j/m}\right| \leqq \text{Const.}\, |\lambda|^{-N/m} \tag{3.2$'$}$$

for $|\lambda| \geqq 1$, $d(\lambda) \geqq |\lambda|^{1-(\theta(x))/m+\varepsilon}$ where the constant in (3.2)$'$ depends on N and ε but is independent of x for x in any compact subset of Ω. In these formulas $(-\lambda)^{-j/m}$ stands for the branch of the power which is positive on the negative axis while $c_j(x)$ are certain C^∞ functions on Ω depending only on the differential operator A. In particular:

$$c_0(x) = (2\pi)^{-n}\rho(x)^{-1} \int_{R^n} [A'(x,\xi) + 1]^{-1} d\xi. \tag{3.3}$$

Before we proceed with the rather long proof of Theorem 3.1 (we shall actually prove a more general result) we shall show how this theorem, when combined with tauberian theorem of Malliavin [13], yields the estimates for the remainder in the asymptotic formula for the spectral function which were mentioned in the introduction. A very simple proof of Malliavin's theorem is due to Pleijel [16] who also gave a slight extension of the theorem. It is the following:

THEOREM (MALLIAVIN). *Let $\sigma(t)$ be a non-decreasing function for $t \geqq 0$ such that $\int_0^\infty (1+t)^{-1} d\sigma(t) < +\infty$. Suppose that*

$$\int_0^\infty (t-\lambda)^{-1} d\sigma(t) - c_0(-\lambda)^\alpha = O(|\lambda|^\beta)$$

as $\lambda \to \infty$ *in the complex plane along the curve*: $|\mathrm{Im}\lambda| = |\lambda|^\gamma$, $\mathrm{Re}\lambda \geqq 0$, *where* $-1 < \beta < \alpha < 0$, $0 < \gamma < 1$; c_0 *some non-negative constant. Then*:

$$\sigma(t) = \frac{\sin \pi(\alpha+1)}{\pi(\alpha+1)} c_0 t^{\alpha+1} + O(t^{\alpha+\gamma}) + O(t^{\beta+1}) \tag{3.4}$$

as $t \to +\infty$.

We shall now prove the following result.

THEOREM 3.2. *Let* $\tilde{A}$ *be a self-adjoint bounded from below operator in* $L_{2,\rho}(\Omega)$ *which is the realization of a* ρ*-formally self-adjoint elliptic differential operator* $A(x,D)$ *of order* m. *Let* $e(t;x,y)$ *be the spectral function of* $\tilde{A}$. *Then*:

$$\begin{aligned} e(t;x,x) &- \left[\rho(x)^{-1}(2\pi)^{-n} \int_{A'(x,\xi)<1} d\xi\right] t^{n/m} \\ &= O\left(t^{(n-\theta(x))/m+\varepsilon}\right) \end{aligned} \tag{3.5}$$

as $t \to +\infty$ *for any* $\varepsilon > 0$, *uniformly in* x *in any compact subset of* Ω, *where* $\theta(x)$ *is the function defined in Theorem* 3.1 ($1/2 \leqq \theta(x) \leqq 1$). *In particular* (3.5) *holds with* $\theta(x)$ *replaced by* $1/2$ *in the general case and with* $\theta(x) = 1$ *if* A' *has constant coefficients.*

Proof. Without loss of generality we may assume that $\tilde{A}$ is positive. Suppose first that $m > n$. Let $R_\lambda(x,y)$ be the resolvent kernel of $\tilde{A}$. By the representation formula (2.6) we have:

$$R_\lambda(x,x) = \int_0^\infty (t-\lambda)^{-1} de(t;x,x).$$

Applying Theorem 3.1, using only the first term in the asymptotic expansion (3.2), we have

$$R_\lambda(x,x) - c_0(x)(-\lambda)^{n/m-1} = O(|\lambda|^{(n-1)/m-1}) \tag{3.6}$$

as $\lambda \to \infty$ along the curve $|\mathrm{Im}\,\lambda| = |\lambda|^{1-(\theta(x))/m+\varepsilon}$, Re $\lambda \geqq 1$, for any $\varepsilon > 0$. We are now in a position to apply Malliavin's tauberian theorem to $\sigma(t) = e(t;x,x)$ (a non-decreasing function) with $\alpha = n/m - 1$, $\beta = (n-1)/m - 1$ and $\gamma = 1 - (\theta(x))/m + \varepsilon$. From (3.4) it follows that

$$e(t;x,x) = \frac{\sin(n\pi/m)}{n\pi/m} c_0(x)\, t^{n/m} + O(t^{(n-\theta(x))/m+\varepsilon}). \tag{3.7}$$

By checking the constants in Pleijel's proof of Malliavin's theorem [16] one also finds (since the O estimate in (3.6) is uniform in x in any compact subset of Ω)

that the O estimate in (3.7) is uniform in x in any compact subset of Ω. A simple computation (using (3.3)) shows that the coefficient of $t^{n/m}$ in (3.7) is the same as the coefficient of $t^{n/m}$ in (3.5). This proves the theorem for $m > n$.

Suppose now that $m \leqq n$. Choose an integer $k > n/m$ and consider the spectral function $e_k(t; x, y)$ of $\tilde{A}^k$. Clearly, $e_k(t; x, y) = e(t^{1/k}; x, y)$. Moreover, $\tilde{A}^k$ is a self-adjoint realization of $\tilde{A}^k$, an elliptic differential operator of order $km > n$. Hence it follows from the special case of the theorem just proved (noting that the function $\theta(x)$ is independent of k) that

$$(3.8) \qquad e(t^{1/k}; x, x) - \left[\rho(x)^{-1}(2\pi)^{-n} \int_{A'(x,\xi)^k<1} d\xi\right] t^{n/km} = O(t^{(n-\theta(x))/km+\varepsilon})$$

uniformly in x in any compact subset of Ω. Replacing $t^{1/k}$ by t in (3.8) we obtain (3.5) and complete the proof.

The remainder of the paper is devoted to the proof of a general asymptotic formula for resolvent kernels containing Theorem 3.1 as a special case.

4. **Preliminary results on fundamental solutions and related kernels.** In this section we consider integral operators acting of functions on R^n. We denote by $H_\infty = H_\infty(R^n)$ the class of functions $u \in C^\infty(R^n)$ such that $D^\alpha u \in L_2(R^n)$ for $|\alpha| \geqq 0$. By $C_*^\infty(R^n)$ we denote the class of functions $u \in C^\infty(R^n)$ such that u and all its derivatives are bounded on R^n. Let s be a real number. The s-norm of $u \in H_\infty$ is defined in the usual way:

$$(4.1) \qquad \|u\|_s^2 = \int_{R^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

Here and in the following $\hat{u}(\xi)$ stands for the Fourier transform:

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{R^n} u(x) e^{-i\xi.x} dx.$$

Let $A(D)$ be a positive elliptic differential operator with constant coefficients of order m and with no lower order terms. It is well known that $A(D)$ has a unique self-adjoint realization in $L_2(R^n)$ which we shall denote by $\tilde{A}$. The operator $\tilde{A}$ is positive and its domain of definition is $H_m(R^n)$. Let $F_\lambda = (\tilde{A} - \lambda)^{-1}$ be the resolvent of $\tilde{A}$ defined for any complex λ not contained in the non-negative axis and denote by F_λ^j its jth power ($j \geqq 1$). As before we denote by $d(\lambda)$ the distance of λ from the positive axis.

LEMMA 4.1. *The operator F_λ^j defines a one to one linear map: $H_\infty \to H_\infty$. For any two real numbers s, t with $s \leqq t \leqq s + jm$ the following inequality holds:*

$$(4.2) \qquad \|F_\lambda^j f\|_t \leqq (3\gamma)^j \frac{|\lambda|^{(t-s)/m}}{d(\lambda)^j} \|f\|_s$$

for $f \in H_\infty$ and $|\lambda| \geqq 1$ where γ is the ellipticity constant:

$$\gamma = \sup_{\xi \in R^n} \frac{(1+|\xi|^2)^{m/2}}{1+A(\xi)}. \tag{4.3}$$

For $t = s$ the constant in (4.2) *can be replaced by* 1.

Proof. By Fourier transformation:

$$\widehat{(F_\lambda^j f)}(\xi) = \frac{\hat{f}(\xi)}{(A(\xi)-\lambda)^j}. \tag{4.4}$$

which implies that F_λ^j yields a one to-one-map: $H_\infty \to H_\infty$. From (4.4) it follows further that

$$\begin{aligned} \|F_\lambda^j f\|_t^2 &= \int_{R^n} \frac{|\hat{f}(\xi)|^2}{|A(\xi)-\lambda|^{2j}}(1+|\xi|^2)^t d\xi \\ &= \int_{R^n} |\hat{f}(\xi)|^2(1+|\xi|^2)^s \cdot \frac{(1+|\xi|^2)^{t-s}}{|A(\xi)-\lambda|^{2j}} d\xi \leqq C_\lambda^2 \|f\|_s^2 \end{aligned} \tag{4.5}$$

where

$$C_\lambda^2 = \sup_{\xi \in R^n} \frac{(1+|\xi|^2)^{t-s}}{|A(\xi)-\lambda|^{2j}}$$

Clearly $C_\lambda = d(\lambda)^{-j}$ for $t = s$. Using the estimate $|A(\xi)-\lambda| \geqq d(\lambda)$ we have for $s < t \leqq s + jm$, $|\lambda| \geqq 1$:

$$\begin{aligned} \frac{(1+|\xi|^2)^{t-s}}{|A(\xi)-\lambda|^{2j}} &\leqq \frac{[\gamma(1+A(\xi))]^{2(t-s)/m}}{|A(\xi)-\lambda|^{2j}} \\ &= \left|\frac{\gamma(1+A(\xi))}{A(\xi)-\lambda}\right|^{2(t-s)/m} \cdot \frac{1}{|A(\xi)-\lambda|^{2j-2(t-s)/m}} \\ &\leqq \gamma^{2(t-s)/m}\left|1+\frac{\lambda+1}{A(\xi)-\lambda}\right|^{2(t-s)/m} d(\lambda)^{-2j+2(t-s)/m} \\ &\leqq \gamma^{2(t-s)/m}\left|1+\frac{|\lambda|+1}{d(\lambda)}\right|^{2(t-s)/m} d(\lambda)^{-2j+2(t-s)/m} \\ &\leqq (3\gamma|\lambda|)^{2(t-s)/m} d(\lambda)^{-2j}. \end{aligned}$$

Hence

$$C_\lambda \leqq (3\gamma)^j |\lambda|^{(t-s)/m} d(\lambda)^{-j} \text{ for } |\lambda| \geqq 1,$$

and combining this estimate with (4.5) we obtain the desired inequality (4.2).

Suppose now that $mj > n$. It follows from (4.4) that the operator F_λ^j is an integral (convolution) operator with a continuous and bounded kernel $F_\lambda^j(x,y) = F_\lambda(x-y,0)$ given by

$$(4.6)\qquad F_\lambda{}^j(x,y) = (2\pi)^{-n}\int_{R^n} \frac{e^{i(x-y)\cdot\xi}}{(A(\xi)-\lambda)^j}\,d\xi.$$

Moreover the kernel (4.6) has continuous bounded derivations up to the order $mj-n-1$ on $R^n \times R^n$. In particular for $y = x$ it follows by a straightforward computation that for $|\alpha| \leqq mj-n-1$:

$$(4.7)\qquad \begin{aligned}(D_x{}^\alpha F_\lambda{}^j)(x,x) &= (D_x{}^\alpha F_\lambda{}^j)(0,0)\\ &= (-\lambda)^{(n+|\alpha|)/m-j}\cdot(2\pi)^{-n}\int_{R^n}\frac{\xi^\alpha}{(A(\xi)+1)^j}\,d\xi,\end{aligned}$$

where $(-\lambda)^{(n+|\alpha|)m-j}$ is the analytic branch of the power in the complex plane cut along the positive axis which is positive on the negative axis.

We consider now an operator S_λ of the form:

$$(4.8)\qquad S_\lambda = B_{k+1}(x,D)F_\lambda{}^{j_k}B_k(x,D)F_\lambda{}^{j_{k-1}}\cdots F_\lambda{}^{j_1}B_1(x,D)$$

where the $B_\nu(x,D)$ are differential operators of orders $l_\nu \geqq 0$, $\nu = 1,\cdots,k+1$, with coefficients belonging to $C_*^\infty(R^n)$. We set:

$$l = \sum_{\nu=1}^{k+1} l_\nu, \qquad j = \sum_{\nu=1}^{k} j_\nu.$$

It is well known that a differential operator B_ν of the above kind defines a bounded linear transformation: $H_s \to H_{s-l_\nu}$ for every real s. This follows from the easily established estimate:

$$(4.9)\qquad \|B_\nu u\|_{s-l_\nu} \leqq C_{s,\nu}\|u\|_s, \qquad u \in H_\infty,$$

where $C_{s,\nu}$ is a constant depending only on s, l_ν, n and on a common bound for the coefficients of B_ν and their derivatives up to a certain order $N = N(s,l_\nu,n)$. From the properties of the operators B_ν and $F_\lambda{}^j$ it is thus clear that S_λ which is a well defined linear operator: $H_\infty \to H_\infty$ is (after completion) a bounded linear operator: $H_s \to H_t$ for any s, t such that $t \leqq s + mj - l$.

By an alternate application of (4.9) and Lemma 4.1 to the factors of S_λ it is easy to see that the following estimates hold for $s - l \leqq t \leqq s + mj - l$:

$$(4.10)\qquad \|S_\lambda f\|_t \leqq (3\gamma)^j C\,\frac{|\lambda|^{(t-s+l)/m}}{d(\lambda)^j}\|f\|_s, \qquad f \in H_\infty,$$

where γ is the ellipticity constant (4.3) and C is a constant depending only on j, l, m, n and on the B_ν (here and in the following when we say that a constant depends on $\{B_\nu\}$ we mean that it depends on a common bound for the coefficients of $\{B_\nu\}$ and their derivatives up to a certain order $N = N(j,l,m,n)$). Indeed it suffices to verify (4.10) for the extreme values $t = s - l$ and $t = s - l + mj$; the result for an intermediate t will then follow from the well known interpolation inequality:

$$\| u \|_t \leqq (\| u \|_{t_1})^{(t_2 - t)/(t_2 - t_1)} (\| u \|_{t_2})^{(t - t_1)/(t_2 - t_1)}, \qquad t_1 < t < t_2 .$$

Now, the estimate (4.10) for $t = s - l + mj$ follows by considering each factor $F_\lambda^{j_\nu}$ of S_λ as a bounded linear operator: $H_r \to H_{r + mj_\nu}$ with norm estimated in Lemma 4.1:

$$\| F_\lambda^{j_\nu} f \|_{r + mj_\nu} \leqq (3\gamma)^{j_\nu} \left(\frac{|\lambda|}{d(\lambda)} \right)^j \| f \|_r . \tag{4.11}$$

Thus, if C denotes a generic constant depending only on the B_ν and on j, l, m, n we have by (4.9) and (4.11):

$$\begin{aligned} \| B_{k+1} F_\lambda^{j_k} B_k \cdots F_\lambda^{j_1} B_1 f \|_{s - l + mj} &\leqq C \| F_\lambda^{j_k} B_k \cdots F_\lambda^{j_1} B_1 f \|_{s - l + l_{k+1} + mj} \\ &\leqq C (3\gamma)^{j_k} \left(\frac{|\lambda|}{d(\lambda)} \right)^{j_k} \| B_k \cdots F_\lambda^{j_1} B_1 f \|_{s - (l - l_{k+1}) + m(j - j_k)} \\ &\leqq \cdots \leqq C (3\gamma)^j \| f \|_s . \end{aligned}$$

Similarly the estimate (4.10) for $t = s - l$ follows considering this time each factor $F_\lambda^{j_\nu}$ of S_λ as a bounded linear operator: $H_r \to H_r$ with norm $d(\lambda)^{-j_\nu}$.

From now on we assume that $mj - l \geqq 0$ and consider S_λ as a bounded linear operator: $L_2(R^n) \to L_2(R^n)$. It is clear that S_λ^* the adjoint of S_λ in $L_2(R^n)$, is an operator of the same type:

$$S_\lambda^* = B_1^* F_{\bar\lambda}^{j_1} \cdots F_{\bar\lambda}^{j_k} B_{k+1}^*$$

where B_ν^* denotes the formal adjoint of B_ν. We have:

THEOREM 4.1. *Suppose that $mj - l > n$. Then S_λ is an integral operator with a continuous bounded kernel $S_\lambda(x, y)$ on $R^n \times R^n$, possessing continuous bounded derivatives up to the order $mj - l - n - 1$, satisfying the following estimate:*

$$| S_\lambda(x, y) | \leqq \gamma^j C_0 \frac{|\lambda|^{(n+l)/m}}{d(\lambda)^j}, \qquad |\lambda| \geqq 1, \tag{4.12}$$

where γ is the ellipticity constant and C_0 is a constant depending only on the B_ν and on j, l, m, n.

Proof. Set $m' = mj - l$. By the preceding remarks S_λ is a bounded linear operator in $L_2(R^n)$ having its range in $H_{m'}(R^n)$. Similarly, range $(S_\lambda^*) \subset H_{m'}(R^n)$. Hence, since $m' > n$ it follows from Theorem 2.1 that S_λ is an integral operator with a continuous and bounded kernel. Moreover, using (4.10) we obtain for the zero and m' norms of S_λ and S_λ^* (def. (2.1)) the estimates:

$$\| S_\lambda \|_0 \leqq (3\gamma)^j \, C \frac{|\lambda|^{l/m}}{d(\lambda)^j}, \quad \| S_\lambda \|_{m'} \leqq (3\gamma)^j C \left(\frac{|\lambda|}{d(\lambda)} \right)^j, \tag{4.13}$$
$$\| S_\lambda^* \|_{m'} \leqq (3\gamma)^j \, C \left(\frac{|\lambda|}{d(\lambda)} \right)^j, \qquad |\lambda| \geqq 1,$$

with a constant C depending only on the B_ν and on j, l, m, n. Applying now the inequality (2.2) to the kernel $S_\lambda(x, y)$ of S_λ, using (4.13), we find:

$$\begin{aligned} |S_\lambda(x, y)| &\leqq \text{Const.} \, (\| S_\lambda \|_{m'} + \| S_\lambda \|_{m'})^{n/m'} \| S_\lambda \|_0^{1-n/m'} \\ &\leqq \gamma^j C_0 \frac{|\lambda|^{(n+l)/m}}{d(\lambda)^j}, \end{aligned}$$

which is the estimate (4.12). Finally, to show that $S_\lambda(x, y)$ is continuously differentiable up to the order $mj - l - n - 1$, consider the operator: $S_\lambda^{\alpha,\beta} = D^\alpha S_\lambda D^\beta$ for any multi-indices α, β with $|\alpha| + |\beta| \leqq mj - l - n - 1$. One checks readily that $S_\lambda^{\alpha,\beta}$ is a bounded linear operator in $L_2(R^n)$ which satisfies the conditions of Theorem 2.1. Hence, $S_\lambda^{\alpha,\beta}$ is an integral operator with a continuous bounded kernel $S_\lambda^{\alpha,\beta}(x, y)$. Now from the definition of $S_\lambda^{\alpha,\beta}$ it follows that in the distribution sense:

$$S_\lambda^{\alpha,\beta}(x, y) = (-1)^{|\beta|} D_x^\alpha D_y^\beta S_\lambda(x, y),$$

which together with the continuity of $S_\lambda^{\alpha,\beta}(x, y)$ imply the existence of the derivatives in the classical sense. This completes the proof.

In addition to S_λ we consider an operator valued function T_λ of the form:

$$T_\lambda = S_\lambda G_\lambda \tag{4.14}$$

where for each complex λ not on the non-negative axis G_λ is a bounded linear operator in $L_2(R^n)$ such that range (G_λ) and range (G_λ^*) are contained in $H_m(R^n)$ and such that:

$$\| G_\lambda \|_0 \leqq \frac{c}{d(\lambda)}, \quad \| G_\lambda \|_m \leqq c \frac{|\lambda|}{d(\lambda)}, \tag{4.15}$$
$$\| G_\lambda^* \|_m \leqq c \frac{|\lambda|}{d(\lambda)} \quad \text{for } |\lambda| \geqq 1,$$

c some constant.

THEOREM 4.2. *Suppose that $m > n$ and $mj \geqq l$. Then T_λ is an integral operator with a continuous bounded kernel $T_\lambda(x, y)$, possessing continuous bounded x-derivatives up to the order $m - n - 1$, such that*

$$|T_\lambda(x, y)| \leqq c\gamma^j C_0 \frac{|\lambda|^{(n+l)/m}}{d(\lambda)^{j+1}} \quad \textit{for } |\lambda| \geqq 1, \tag{4.16}$$

where c is the constant in (4.15) *and* C_0 *is a constant having the same dependence as in* (4.12).

Proof. Since S_λ is also a bounded linear operator: $H_m \to H_m$ it follows from (4.14) and the properties of G_λ that range (T_λ) and range (T_λ^*) are contained in $H_m(R^n)$. Hence, since $m > n$, it follows from Theorem 2.1 that T_λ is an integral operator with a continuous and bounded kernel $T_\lambda(x, y)$. Now, using (4.14), (4.10) and (4.15) we have:

$$\begin{aligned} \| T_\lambda \|_0 &\leqq \| S_\lambda \|_0 \cdot \| G_\lambda \|_0 \leqq (3\gamma)^j C \frac{|\lambda|^{l/m}}{d(\lambda)^j} \cdot \frac{c}{d(\lambda)} \\ &= c(3\gamma)^j C \frac{|\lambda|^{l/m}}{d(\lambda)^{j+1}} . \end{aligned} \tag{4.17}$$

$$\begin{aligned} \| T_\lambda^* \|_m &\leqq \| G_\lambda^* \|_m \cdot \| S_\lambda^* \|_0 \leqq c\frac{|\lambda|}{d(\lambda)} \cdot (3\gamma)^j C \frac{|\lambda|^{l/m}}{d(\lambda)^j} \\ &= c(3\gamma)^j C \frac{|\lambda|^{1+l/m}}{d(\lambda)^{j+1}} . \end{aligned} \tag{4.17$'$}$$

Also, since for every $f \in L_2(R^n)$:

$$\| T_\lambda f \|_m = \| S_\lambda G_\lambda f \|_m \leqq (3\gamma)^j C \frac{|\lambda|^{l/m}}{d(\lambda)^j} \| G_\lambda f \|_m,$$

(using (4.10) for $t = s = m$) we have:

$$\| T_\lambda \|_m \leqq (3\gamma)^j C \frac{|\lambda|^{l/m}}{d(\lambda)^j} \| G_\lambda \|_m \leqq c(3\gamma)^j C \frac{|\lambda|^{1+l/m}}{d(\lambda)^{j+1}} . \tag{4.17$''$}$$

Combining now (4.17), (4.17)$'$ and (4.17)$''$ with the inequality (2.2) applied to the kernel $T_\lambda(x, y)$ we arrive at the estimate (4.16). Finally the proof of the differentiability of the kernel $T_\lambda(x, y)$ is very much the same as the proof of the differentiability of the kernel $S_\lambda(x, y)$ given above. We omit the details.

5. **Some properties of commutators.** In this section we shall prove some results for multiple commutators of operators which will be needed later on. Although the case of interest to us is that of differential operators we shall start by considering a more general situation. Let M be a linear space over a field K and let $A, B: M \to M$ be linear operators. Denote by $S(r, t)$ the set of r-vectors $J = (j_1, \cdots, j_r)$ with integral components $0 \leqq j_i \leqq t$, $i = 1, \cdots, r$. (The elements of $S(r, t)$ are multi-indices in R^n; to avoid confusion we use here Latin and not Greek letters). Set $|J| = j_1 + \cdots + j_r$, $S(r) = \bigcup_{t=1}^{\infty} S(r, t)$ and denote by $J \cup (j_{r+1})$ $(J \in S(r))$ the vector $(j_1, \cdots, j_r, j_{r+1}) \in S(r+1)$. Define a zero dimensional vector (belonging to $S(o, t)$) to be the empty vector. For the empty vector $J = \varnothing$ set: $|J| = 0$ and $\varnothing \cup (j_1) = (j_1)$.

We shall now define inductively multiple commutators $[B, A; J]$, J non-empty, in the following way:

$$[B, A; (0)] = B \tag{5.1}$$

$$[B, A; (j+1)] = [B, A; (j)]A - A[B, A; (j)] \tag{5.2}$$

$$[B, A; J \cup (j_{r+1})] = [B[B, A; J], A; (j_{r+1})] \tag{5.3}$$

(Note that $[B, A; (1)] = BA - AB$ is the usual commutator of B and A).

Let $\lambda \in K$ be such that $A - \lambda = A - \lambda I$ is one-one and onto, and set $F_\lambda = (A - \lambda)^{-1}$.

THEOREM 5.1. *Let r and k be positive integers. Then*

$$\begin{aligned}(F_\lambda B)^r = &\sum_{J \in S(r,k-1)} [B, A; J]\, F_\lambda^{|J|+r} \\ &+ \sum_{s=0}^{r-1} (F_\lambda B)^s F_\lambda \sum_{J \in S(r-s-1,k-1)} [B, A; J \cup (k)] F_\lambda^{|J|+k+r-s-1}\end{aligned} \tag{5.4}$$

Proof. We shall proceed inductively in several steps. Consider first the case $r = k = 1$. Formula (5.4) reduces then to

$$F_\lambda B = BF_\lambda + F_\lambda(BA - AB)F_\lambda \tag{5.5}$$

which is immediately verified by applying $A - \lambda$ on both sides of (5.5) from the left.

Suppose that (5.4) has been established already for $r = 1$ and some k, i.e., suppose that

$$F_\lambda B = \sum_{j=0}^{k-1} [B, A; (j)] F_\lambda^{j+1} + F_\lambda[B, A; (k)] F_\lambda^k \tag{5.6}$$

is true. Using (5.5) with $[B, A; (k)]$ replacing B we find that

$$\begin{aligned}F_\lambda[B, A; (k)] &= [B, A; (k)]F_\lambda + F_\lambda([B, A; (k)]A - A[B, A; (k)])F_\lambda \\ &= [B, A; (k)]F_\lambda + F_\lambda[B, A; (k+1)]F_\lambda\ .\end{aligned} \tag{5.7}$$

Inserting (5.7) in (5.6) we see that

$$\begin{aligned}F_\lambda B &= \sum_{j=0}^{k-1} [B, A; (j)]F_\lambda^{j+1} + [B, A; (k)]F_\lambda^{k+1} + F_\lambda[B, A; (k+1)]F_\lambda^{k+1} \\ &= \sum_{j=0}^{k} [B, A; (j)]F_\lambda^{j+1} + F_\lambda[B, A; (k+1)]F_\lambda^{k+1}\ .\end{aligned}$$

Thus (5.4) is proved for $r = 1$. Assume now that the theorem has been proved for some r. Then

$$(5.8)\qquad \begin{aligned}(F_\lambda B)^{r+1} &= F_\lambda B(F_\lambda B)^r = F_\lambda B \sum_{J\in S(r,k-1)} [B,A;J]F_\lambda^{|J|+r}\\ &+ \sum_{s=0}^{r-1} (F_\lambda B)^{s+1}F_\lambda \sum_{J\in S(r-s-1,k-1)} [B,A;J\cup(k)]F_\lambda^{|J|+k+r-s-1}\ .\end{aligned}$$

According to (5.6) with B replaced by $B[B,A;J]$ we may write the first sum in (5.8) in the form

$$\begin{aligned}F_\lambda B &\sum_{J\in S(r,k-1)} [B,A;J]F_\lambda^{|J|+r}\\ &= \sum_{j=0}^{k-1} \sum_{J\in S(r,k-1)} [B[B,A;J],A;(j)]F_\lambda^{|J|+r+j+1}\\ &\quad + \sum_{J\in S(r,k-1)} F_\lambda[B[B,A;J],A;(k)]F_\lambda^{|J|+r+k}\ .\end{aligned}$$

According to (5.3) this is equal to

$$\sum_{J\in S(r+1,k-1)} [B,A;J]F_\lambda^{|J|+r+1} + \sum_{J\in S(r,k-1)} F_\lambda[B,A;J\cup(k)]F_\lambda^{|J|+r+k}\ .$$

Inserting this in (5.8) we get

$$(5.9)\qquad \begin{aligned}(F_\lambda B)^{r+1} &= \sum_{J\in S(r+1,k-1)} [B,A;J]F_\lambda^{|J|+r+1}\\ &+ \sum_{J\in S(r,k-1)} F_\lambda[B,A;J\cup(k)]F_\lambda^{|J|+k+r}\\ &+ \sum_{s=0}^{r-1} (F_\lambda B)^{s+1}F_\lambda \sum_{J\in S(r-s-1,k-1)} [B,A;J\cup(k)]F_\lambda^{|J|+k+r-s-1}\\ &= \sum_{J\in S(r+1,k-1)} [B,A;J]F_\lambda^{|J|+r+1}\\ &+ \sum_{s=0}^{r} (F_\lambda B)^{s}F_\lambda \sum_{J\in S(r-s,k-1)} [B,A;J\cup(k)]F_\lambda^{|J|+k+r-s}\ .\end{aligned}$$

But (5.9) is (5.4) with $r+1$ instead of r, so that the theorem is proved.

Assume now that there exists a subring $\mathscr{R}$ of the ring of linear transformations from M to M and a function o from $\mathscr{R}$ to the real line so that the following conditions hold:

$$(5.10)\qquad o(0) = -\infty$$

$$(5.11)\qquad o(AB) \leqq o(A) + o(B)$$

$$(5.12)\qquad o(AB - BA) \leqq o(A) + o(B) - 1$$

$$(5.13)\qquad o(I) = 0\ .$$

$o(A)$ is called the order of A.

LEMMA 5.1. *Let $J \in S(r)$, $r > 0$. Then*

$$o[B,A;J] \leqq |J|o(A) + r\,o(B) - |J| \ . \tag{5.14}$$

Proof. If $r = 1$ then (5.14) reads:

$$o([B,A;(j)]) \leqq j\,o(A) + o(B) - j \ . \tag{5.15}$$

When $j = 0$, (5.15) follows from (5.1). Assume that (5.15) has been proved for a certain j. Using (5.2), (5.12), and the induction hypothesis, we find that

$$\begin{aligned} o([B,A;(j+1)]) &= o([B,A;(j)]A - A[B,A;(j)]) \leqq o([B,A;(j)]) + o(A) - 1 \\ &\leqq j\,o(A) + o(B) - j + o(A) - 1 \\ &= (j+1)\,o(A) + o(B) - (j+1) \end{aligned}$$

which proves the theorem in the case $r = 1$. Assuming that the theorem has been proved for a certain r, we get from (5.3), (5.11) and (5.15) that

$$\begin{aligned} o([B,A;J \cup (j_{r+1})]) = o([[B,A;J],A;(j_{r+1})]) &\leqq j_{r+1}o(A) + o(B[B,A;J]) - j_{r+1} \\ &\leqq j_{r+1}o(A) + o(B) + o([B,A;J]) - j_{r+1} \\ &\leqq j_{r+1}o(A) + o(B) + |J|\,o(A) + ro(B) - |J| - j_{r+1} \\ &= |J \cup (j_{r+1})|\,o(A) + (r+1)\,o(B) - |J \cup (j_{r+1})| \end{aligned}$$

and thus the theorem is proved for every r.

LEMMA 5.1 will be applied in the sequel to the case where M is the linear space $H_\infty(R^n)$ and $\mathscr{R}$ is the ring of differential operators with C_*^∞ coefficients. In this case we denote by $o(A)$ the usual order of the differential operator A. We conclude this section by establishing for commutators of differential operators a result which we shall need later on. In this connection let us agree to say that a C^∞ function $u(x)$ has a zero of type p at a point x^0 ($p > 0$ an integer) if u and all its derivatives up to the order $p-1$ vanish at x^0.

THEOREM 5.2. *Let $[B,A;J]$ be a multiple commutator of two differential operators B, A ($J = j(_1,\cdots,j_r)$). Suppose that the coefficients of the principal part of B possess a zero of type $p > |J|/r$ at some point $x^0 \in R^n$. Put $N_J = |J|(o(A)-1) + ro(B)$ (so that $o([B,A,;J]) \leqq N_J$ by (5.14)) and write*:

$$[B,A;J] = \sum_{|\alpha| \leqq N_J} b_{\alpha,J}(x)D^\alpha. \tag{5.16}$$

Then each coefficient $b_{\alpha,J}$ for $N_J - r + |J|/p < |\alpha| \leqq N_J$ has a zero of type $(r + |\alpha| - N_J)p - |J|$ at x^0.

Proof. For the purpose of the proof it is convenient to agree that for any integer $q \leqq 0$ the statement: "$u(x)$ has at a point x^0 a zero of type q" is a true statement which holds in the empty sense. With this convention we have to prove that each coefficient $b_{\alpha,J}$ has a zero of type $(r+|\alpha|-N_J)p-|J|$ at x^0. We shall prove this using a double induction on J. We shall first show that if the theorem is true for $J=(j_1,\cdots,j_r)$ then it is also true for $J'=(j_1,\cdots,j_{r-1},j_r+1)$. Indeed, let $A=\sum a_\alpha D^\alpha$. By definition, using (5.1) – (5.3):

$$\begin{aligned}\sum_{|\alpha|\leqq N_{J'}} b_{\alpha,J'}(x)D^\alpha=[B,A;J'] &= [B,A;J]A-A[B,A;J]\\ &= \sum_{|\beta|\leqq N_J}\ \sum_{|\gamma|\leqq o(A)} [b_{\beta,J}D^\beta,a_\gamma D^\gamma;(1)].\end{aligned}\tag{5.17}$$

From (5.17) it is clear that the coefficient $b_{\alpha,J'}$ is a linear combination of terms of the form:

$$b_{\beta,J}D^{\beta'}a_\gamma \quad \text{or} \quad a_\gamma D^{\gamma'}b_{\beta,J}\tag{5.18}$$

with $|\beta|+|\gamma|-1\geqq|\alpha|$ where β' is a multi-index such that $\beta=\beta'+\beta''$ and $\alpha=\beta''+\gamma$, β'' some complementary multi-index, and similarly γ' is a multi-index such that $\gamma=\gamma'+\gamma''$, $\alpha=\beta+\gamma''$ for some γ''. Note that these restrictions imply that $|\beta'|\geqq 1$, $|\gamma'|\geqq 1$ and that (in the last case):

$$|\beta|=|\alpha|-|\gamma''|=|\alpha|+|\gamma'|-|\gamma|.$$

To prove the theorem for J' it will suffice to show that each of the terms (5.18) has at x^0 a zero of type $(r+|\alpha|-N_{J'})p-|J'|$. Now by the induction assumption the coefficient $b_{\beta,J}$ has at x^0 a zero of type $(r+|\beta|-N_J)p-|J|$. Hence, since $|\beta|\geqq|\alpha|+1-|\gamma|\geqq|\alpha|+1-o(A)$ and $N_{J'}=N_J+o(A)-1$, $|J'|=|J|+1$, we find that a term $b_{\beta,J}D^{\beta'}a_\gamma$ has at x^0 a zero of type:

$$\begin{aligned}(r+|\beta|-N_J)p-|J| &\geqq (r+|\alpha|+1-o(A)-N_J)p-|J|\\ &> (r+|\alpha|-N_{J'})p-|J'|.\end{aligned}$$

Similarly, since $|\beta|=|\alpha|+|\gamma'|-|\gamma|\geqq|\alpha|+|\gamma'|-o(A)$ and $|\gamma'|\geqq 1$, we find that the term $a_\gamma D^{\gamma'}b_{\beta,J}$ has at x^0 a zero of type:

$$\begin{aligned}&(r+|\beta|-N_J)p-|J|-|\gamma'|\\ &\qquad\geqq (r+|\alpha|+|\gamma'|-o(A)-N_J)p-|J|-|\gamma'|\\ &\qquad= (r+|\alpha|+1-o(A)-N_J)p-(|J|+1)+(p-1)(|\gamma'|-1)\\ &\qquad\geqq (r+|\alpha|-N_{J'})p-|J'|.\end{aligned}$$

These computations show that the theorem holds for J' as claimed.

We shall complete the proof by induction on r. Suppose first that $r=1$. By the result just proved in order to establish the theorem for $J=(j_1)$ it suffices to show that the theorem holds for $J=(0)$. This, however, is trivial since $[B,A;(0)]=B$ and one checks readily that the statement of the theorem in this case reduces to our assumption on the coefficients of B. Hence the theorem holds or $r = 1$. Next assume that the theorem holds for some r. We shall show that it holds for $r+1$. Using again the result established above it will suffice to prove the theorem for $J^0 = J \cup (0)$ where $J = (j_1, \cdots, j_r)$. Now, letting $B = \sum b_\alpha D^\alpha$, we have:

$$\begin{aligned} \sum b_{\alpha,J^0} D^\alpha &= [B, A; J^0] = B[B, A; J] \\ &= \sum_\beta \sum_\gamma b_\beta D^\beta (b_{\gamma,J} D^\gamma). \end{aligned} \tag{5.19}$$

From (5.19) it follows that b_{α,J^0} is a linear combination of terms of the type:

$$b_\beta D^{\beta'} b_{\gamma,J} \tag{5.20}$$

with $|\beta| \leqq o(B)$, $|\gamma| \leqq N_J$ and where β' is a multi-index such that $\beta = \beta' + \beta''$, $\beta'' + \gamma = \alpha$ for a certain complementary multi-index β''. Note in particular that these relations imply: $|\gamma| = |\alpha| + |\beta'| - |\beta|$. Consider a typical term (5.20) and assume first that $|\beta| \leqq o(B) - 1$. Using the induction assumption together with the estimate $|\gamma| \geqq |\alpha| + |\beta'| - o(B) + 1$, we find that $D^{\beta'} b_{\gamma,J}$ has at x^0 a zero of type:

$$\begin{aligned} (r + |\gamma| - N_J)p - |J| - |\beta'| &\geqq (r + |\alpha| + |\beta'| - o(B) + 1 - N_J)p - |J| - |\beta'| \\ = (r + 1 + |\alpha| - N_{J^0})p - |J^0| + (p-1)|\beta'| &\geqq (r + 1 + |\alpha| - N_{J^0})p - |J^0|, \end{aligned}$$

(using also that $N_{J^0} = N_J + o(B)$ and $|J^0| = |J|$).

Next suppose that $|\beta| = m$. In this case b_β has a zero of type p at x^0. Using this, the induction assumption and the estimate: $|\gamma| \geqq |\alpha| + |\beta'| - o(B)$, it follows that $b_\beta D^{\beta'} b_{\gamma,J}$ has at x^0 a zero of type:

$$\begin{aligned} p + (r &+ |\gamma| - N_J)p - |J| - |\beta'| \\ &\geqq (1 + r + |\alpha| + |\beta'| - o(B) - N_J)p - |J| - |\beta'| \\ &\geqq (r + 1 + |\alpha| - N_{J^0})p - |J^0|. \end{aligned}$$

The above computations show that b_{γ,J^0} has at x^0 a zero of type

$$(r + 1 + |\alpha| - N_{J^0})p - |J^0|,$$

which is the desired result for $r + 1$. This completes the proof.

6. **The asymptotic expansion of resolvent kernels.** We shall first discuss a class of operators on R^n. Let $A(x, D)$ be a positive elliptic differential operator on R^n, ρ-formally self-adjoint and of order $m > n$. We assume that the coefficients of A are in $C_*^\infty(R^n)$, that $\rho \in C_*^\infty(R^n)$ and that $\rho(x) \geqq \delta > o$, δ some constant. We also assume that A is uniformly elliptic: $A'(x, \xi) \geqq C|\xi|^m$ for x and ξ in R^n, C a positive constant. Considering A as a symmetric operator in the Hilbert space $L_{2,\rho}(R^n)$ with domain $C_0^\infty(R^n)$ we denote its closure by $\tilde{A}$. It is well known that $\tilde{A}$ is a self-adjoint operator with domain of definition $H_m(R^n)$. Moreover, $\tilde{A}$ is the unique

self-adjoint realization of A in $L_{2,\rho}(R^n)$. All these facts follow easily from the a-priori estimate:

$$\| u \|_m \leqq \text{Const.}(\| Au \|_0 + \| u \|_0)$$

which holds for $u \in H_m(R^n)$ and from the regularity theory of weak solutions elliptic equations (e.g. [1]). In addition it follows from Gårding's inequality that $\tilde{A}$ is bounded from below. In the following we shall assume without loss of generality that $\tilde{A}$ is positive.

Consider now the resolvent operator $R_\lambda = (\tilde{A} - \lambda)^{-1}$. From our previous discussion it follows that R_λ is an integral operator in $L_{2,\rho}(R^n)$ with continuous and bounded kernel $R_\lambda(x,y)$. Since $L_{2,\rho}(R^n)$ and $L_2(R^n)$ are the same function spaces on which two equivalent Hilbert norms are defined, we may consider $\tilde{A}$ and R_λ as operators in $L_2(R_n)$. We shall denote by G_λ the resolvent operator R_λ when considered as an operator in $L_2(R^n)$. It is an integral operator with a kernel: $G_\lambda(x,y) = R_\lambda(x,y)\rho(y)$. The operator $G_\lambda : L_2 \to L_2$ can also be considered as an operator: $L_2 \to H_m$. We have the following norm estimates:

$$\text{(6.1)} \qquad \begin{aligned} &\| G_\lambda \|_0 \leqq cd(\lambda)^{-1}, \quad \| G_\lambda \|_m \leqq c \frac{|\lambda|}{d(\lambda)}, \\ &\| G_\lambda^* \|_m \leqq c \frac{|\lambda|}{d(\lambda)} \text{ for } |\lambda| \geqq 1, \end{aligned}$$

c a constant, where as before $d(\lambda)$ denotes the distance of λ from the positive axis. Indeed the first inequality is immediate since $\tilde{A}$ is a positive operator in $L_{2,\rho}(R^n)$. To derive the second inequality write G_λ in the form: $G_\lambda = G_{-1} U_\lambda$ where $U_\lambda = (\tilde{A} + 1)((\tilde{A} - \lambda)^{-1}$. Clearly U_λ is a bounded operator in L_2 whose norm when considered as an operator: $L_{2,\rho} \to L_{2,\rho}$ does not exceed

$$\sup_{-\infty < t < \infty} \frac{|t+1|}{|t-\lambda|} \leqq 1 + \frac{|\lambda|+1}{d(\lambda)} \leqq 3 \frac{|\lambda|}{d(\lambda)} \text{ for } |\lambda| \geqq 1.$$

Hence:

$$\| G_\lambda \|_m \leqq \| G_{-1} \|_m \| U_\lambda \|_0 \leqq c \frac{|\lambda|}{d(\lambda)}.$$

The last estimate in (6.1) follows of course from the second noting that $G_\lambda^* = \rho G_{\bar{\lambda}} \rho^{-1}$.

We proceed now to derive the asymptotic expansion of $G_\lambda(x,y)$. To this end we fix an arbitrary point x^0 in Ω and set: $A_0(D) = A'(x^0, D)$, $B(x,D) = A'(x_0, D) - A(x,D)$. As in section 4 we denote by $\tilde{A}_0$ the unique self-adjoint realization of A_0 in $L_2(R^n)$ and by F_λ the resolvent of A_0, $F_\lambda = (\tilde{A}_0 - \lambda)^{-1}$. Let $f \in L_2(R^n)$ and set $u = G_\lambda f$. We have:

$$(A_0 - \lambda)u = (A - \lambda)u + Bu = f + Bu,$$

so that

$$u = F_\lambda f + F_\lambda B u,$$

or equivalently

$$G_\lambda = F_\lambda + F_\lambda B G_\lambda.$$

Hence

$$(6.2)\qquad \begin{aligned} G_\lambda &= F_\lambda + F_\lambda B G_\lambda = F_\lambda + F_\lambda B F_\lambda + (F_\lambda B)^2 G_\lambda = \cdots = \\ &= \sum_{r=0}^{l-1} (F_\lambda B)^r F_\lambda + (F_\lambda B)^l G_\lambda \end{aligned}$$

for every integer $l \geqq 1$.

Considering A_0, B and F_λ as linear operators: $H_\infty \to H_\infty$ (note that $A_0 - \lambda$ is one-to-one from H_∞ onto itself) we apply Theorem 5.1 to $(F_\lambda B)^r$. After completion in $L_2(R^n)$ it follows from (5.4) and from (6.2) that

$$(6.3)\qquad \begin{aligned} G_\lambda = F_\lambda &+ \sum_{r=1}^{l-1} \sum_{J \in S(r,k-1)} [B, A_0; J] F_\lambda^{|J|+r+1} \\ &+ \sum_{r=1}^{l-1} \sum_{s=0}^{r-1} (F_\lambda B)^s F_\lambda \sum_{J \in S(r-s-1,k-1)} [B, A_0; J \cup (k)] F_\lambda^{|J|+k+r-s} \\ &+ (F_\lambda B)^l G_\lambda, \end{aligned}$$

where k is an arbitrary positive integer.

According to Lemma 5.1 the order of the differential operator $[B, A_0; J]$ for $J \in S(r)$ is at most: $|J|(m-1) + r o(B)$. If A' has constant coefficients $o(B) \leqq m-1$ and $o([B, A_0; J]) \leqq (|J| + r)(m-1)$. In the general case: $o([B, A_0; J]) \leqq (|J| + r)(m-1) + r$. Consider the right hand side of (6.3). Clearly F_λ is an integral operator with a continuous and bounded kernel $F_\lambda(x,y) = F_\lambda(x-y,0)$. Using the results of section 4 and our estimate on the order of $[B, A_0; J]$ it follows that every term $[B, A_0; J]F_\lambda^{|J|+r+1}$ which appears in the first sum in (6.3) is an integral operator with a continuous and bounded kernel

$$([B, A; J] F_\lambda^{|J|+r+1})(x,y).$$

Set:

$$(6.4)\qquad \begin{aligned} H_\lambda^{k,l}(x,y) &= G_\lambda(x,y) - F_\lambda(x,y) \\ &\quad - \sum_{r=1}^{l-1} \sum_{J \in S(r,k-1)} ([B, A_0; J] F_\lambda^{|J|+r+1})(x,y). \end{aligned}$$

Then $H_\lambda^{k,l}(x,y)$ is the kernel of the operator given by the sum of the two last members of (6.3).

Our object is to estimate $H_\lambda^{k,l}(x,y)$. To this end consider first the operator given by the one before last member of (6.3). It is a sum of operators:

$$(F_\lambda B)^s F_\lambda[B, A_0; J \cup (k)] F_\lambda{}^{|J|+k+r-s} \tag{6.5}$$

with $1 \leqq r \leqq l-1$, $0 \leqq s \leqq r-1$. $J \in S(r-s-1, k-1)$. According to Theorem 4.1 the operator (6.5) is an integral operator with a continuous and bounded kernel such that (since $o([B, A_0; J \cup (k)] \leqq (|J|+k)(m-1)+(r-s)\, o(B))$:

$$\begin{aligned} &|((F_\lambda B)^s F_\lambda[B, A_0; J \cup (k)] F_\lambda^{|J|+k+r-s})(x,y)| \\ &\qquad \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^{|J|+k+r} \cdot |\lambda|^{r(o(B)-m+1)/m} \end{aligned} \tag{6.6}$$

for $|\lambda| \geqq 1$. Here and in the following C denotes a generic constant which is independent of λ, x, y and x^0 (C depends however on k and l).

In particular it follows from (6.6) that when $o(B) < m$:

$$\begin{aligned} &|((F_\lambda B)^s F_\lambda[B, A_0; J \cup (k)] F_\lambda^{|J|+k+r-s})(x,y)| \\ &\qquad \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^{k+1} \end{aligned} \tag{6.7}$$

for $d(\lambda) \geqq |\lambda|^{1-1/m}$, $|\lambda| \geqq 1$.

If $o(B) = m$ it follows from (6.6) that if $k \geqq l + (l-1)/\varepsilon m$ for some fixed $\varepsilon > 0$, then:

$$\begin{aligned} &|((F_\lambda B)^s F_\lambda[B, A_0; J \cup (k)] F_\lambda^{|J|+k+r-s})(x,y)| \\ &\qquad \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{(d\lambda)}\right)^{l} \end{aligned} \tag{6.7$'$}$$

for $d(\lambda) \geqq |\lambda|^{1-1/m+\varepsilon}$, $|\lambda| \geqq 1$.

Consider now the last member of (6.3). Since G_λ verifies (6.1) it follows from Theorem 4.2 that $(F_\lambda B)^l G_\lambda$ is an integral operator with a continuous and bounded kernel such that

$$|((F_\lambda B)^l G_\lambda)(x,y)| \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{o(B)/m}}{d(\lambda)}\right)^{l}, \quad |\lambda| \geqq 1. \tag{6.8}$$

Suppose that $o(B) < m$. From (6.8). (6.7), (6.3) and (6.4) we obtain for $H_\lambda^{k,l}$ the following estimate when $k \geqq l-1$:

$$H_\lambda^{k,l}(x,y) \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^{l} \tag{6.9}$$

for $d(\lambda) \geqq |\lambda|^{1-1/m}$, $|\lambda| \geqq 1$.

When $o(B)=m$ the estimate (6.8) does not give us the information we look for. In this case, however, we shall show that (6.8) can be replaced by a better estimate if x is restricted to a small neighborhood of x^0 which depends on λ. To this end apply formula (5.4) to $(F_\lambda B)^l$ and write $(F_\lambda B)^l G_\lambda$ in the form:

$$(6.10)\qquad \begin{aligned}(F_\lambda B)^l G_\lambda &= \sum_{J\in S(l,q-1)} [B,A_0;J]F_\lambda^{|J|+l}G_\lambda \\ &+ \sum_{s=0}^{l-1} (F_\lambda B)^s F_\lambda \sum_{J\in S(l-s-1,q-1)} [B,A_0;J\cup(q)]F_\lambda^{|J|+q+l-s-1}G_\lambda,\end{aligned}$$

where $q \geqq 1$ is an integer to be fixed later on. Consider a typical term in the first sum on the right hand side of (6.10). According to Theorem 4.2 $[B,A_0;J]F_\lambda^{|J|+l}G_\lambda$ is an integral operator with a continuous and bounded kernel

$$([B,A_0;J]F_\lambda^{|J|+l}G_\lambda)(x,y).$$

Write:

$$(6.11)\qquad [B,A_0;J] = \sum_{|\alpha|\leqq N_J} b_{\alpha,J}(x;x^0)D_x^\alpha, \qquad J\in S(l),$$

where $N_J = |J|(m-1)+lm$ (note that the $b_{\alpha,J}$ are C_*^∞ functions in x and x^0). By our definition of B it is clear that the coefficients of its principal part B' vanish at x^0. We shall denote by $p=p(x^0)$ the largest integer $\geqq 1$ such that all the coefficients of B' possess a zero of type p at x^0. If no such largest integer exists, i.e. if all the coefficients of B' possess a zero of infinite order at x^0, we let $p=+\infty$. We set:

$$(6.12)\qquad \theta=\theta(x^0)=\frac{p}{p+1} \qquad \left(\frac{1}{2}\leqq\theta\leqq 1\right),$$

and

$$M_J = M_J(x^0) = \min\left\{N_J, N_J - l + \frac{|J|}{p}\right\}, \qquad J\in S(l).$$

By Theorem 5.2 the coefficients $b_{\alpha,J}$ in (6.11) vanish at $x=x^0$ for $|\alpha|>M_J$. Observe that

$$(6.12)'\qquad \begin{aligned}M_J &\leqq \frac{1}{p+1}N_J + \frac{p}{p+1}\left(N_J - l + \frac{|J|}{p}\right)\\ &= (|J|+l)(m-1)+l-\frac{p}{p+1}l+\frac{1}{p+1}|J| = (|J|+l)(m-\theta).\end{aligned}$$

We proceed now to estimate the kernel:

$$(6.13)\qquad ([B,A_0;J]F_\lambda^{|J|+l}G_\lambda)(x,y) = \sum_{|\alpha|\leqq N_J} b_{\alpha,J}(x,x^0)(D^\alpha F_\lambda^{|J|+l}G_\lambda)(x,y)$$

with $J\in S(l,q-1)$. Consider first a term in the last sum with

$$|\alpha|\leqq M_J \leqq (|J|+l)(m-\theta)$$

(by (6.12)′). Applying Theorem 4.2 to the operator $D^\alpha F_\lambda{}^{|J|+k} G_\lambda$ it follows from (4.16) that

$$\begin{aligned} |b_{\alpha,J}(D^\alpha F_\lambda{}^{|J|+l} G_\lambda)(x,y)| &\leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \frac{|\lambda|^{|\alpha|/m}}{d(\lambda)^{|J|+l}} \\ &\leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{|J|+l} \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l} \end{aligned} \tag{6.14}$$

for $d(\lambda) \geqq |\lambda|^{1-\theta/m}$, $|\lambda| \geqq 1$. Next, if $M_J < N_J$, consider a term in the sum (6.13) with $|\alpha| > M_J$. By Theorem 5.2 the coefficient $b_{\alpha,J}$ (as a function of x) has at x^0 a zero of type $(|\alpha| - M_J)p$. Restrict x to a neighborhood:

$$|x - x^0| \leqq |\lambda|^{-1/p'm} \tag{6.15}$$

where we set

$$p' = p'(x^0) = \min\{p, Q\}, \tag{6.15$'$}$$

Q being an arbitrary but fixed integer > 1 (independent of x^0). Clearly for such x:

$$|b_{\alpha,J}| \leqq C|\lambda|^{(M_J - |\alpha|)/m} . \tag{6.16}$$

Apply now Theorem 4.2 to the operator $D^\alpha F_\lambda{}^{|J|+l} G_\lambda$. It follows from (4.16), (6.16) and (6.12)′ that

$$\begin{aligned} |b_{\alpha,J}(D^\alpha F_\lambda{}^{|J|+l} G_\lambda)(x,y)| &\leqq C|\lambda|^{(M_J-|\alpha|)/m} \frac{|\lambda|^{(n+|\alpha|)/m}}{d(\lambda)^{|J|+l+1}} \\ &\leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{|J|+l} \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l} \end{aligned} \tag{6.17}$$

for $d(\lambda) \geqq |\lambda|^{1-\theta/m}$, $|\lambda| \geqq 1$. Combining (6.13), (6.14) and (6.17) we conclude that

$$|([B, A_0; J] F_\lambda^{|J|+l} G_\lambda)(x,y)| \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l} \tag{6.18}$$

for $d(\lambda) \geqq |\lambda|^{1-\theta/m}$, $\lambda \geqq 1$ and x satisfying (6.15).

Apply now Theorem 4.2 to the operator:

$$(F_\lambda B)^s F_\lambda [B, A_0; J \cup (q)] F_\lambda{}^{|J|+q+l-s-1} G_\lambda, \tag{6.19}$$

with $J \in S(l-s-1,\ q-1)$, $s \leqq l-1$. Since

$$o([B, A_0; J \cup (q)] \leqq (|J| + q)(m-1) + m(l-s),$$

it follows from Theorem 4.2 that (6.19) is an integral operator with a continuous and bounded kernel satisfying:

$$(6.20)\qquad \begin{aligned} &|((F_\lambda B)^s F_\lambda[B, A_0; J \cup (q)]F_\lambda^{|J|+q+l-s-1}G_\lambda)(x,y)| \\ &\qquad \leqq C\,\frac{|\lambda|^{n/m}}{d(\lambda)}\left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^{|J|+q+l}|\lambda|^{l/m}, \quad |\lambda| \geqq 1. \end{aligned}$$

For any given $\varepsilon > o$ we choose now q in (6.20) as the smallest integer $q > l/m\varepsilon$. With this choice we have:

$$(6.20)'\qquad \begin{aligned} &|((F_\lambda B)^s F_\lambda[B, A_0; J \cup (q)]F_\lambda^{|J|+q+l-s-1}G_\lambda)(x,y)| \\ &\qquad \leqq C\,\frac{|\lambda|^{n/m}}{d(\lambda)}\cdot\left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^{l} \end{aligned}$$

for $d(\lambda) \geqq |\lambda|^{1-1/m+\varepsilon}$, $|\lambda| \geqq 1$.

Combining now (6.10), (6.18) and (6.20)′ we find that

$$(6.21)\qquad |((F_\lambda B)^l G_\lambda)(x,y)| \leqq C\frac{|\lambda|^{n/m}}{d(\lambda)}\left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l}$$

for $d(\lambda) \geqq \max\{|\lambda|^{1-\theta/m}, |\lambda|^{1-1/m+\varepsilon}\}$, $|\lambda| \geqq 1$, and x verifying (6.15). Finally, from (6.3), (6.4), (6.7)′ and (6.21) we find that when $o(B) = m$ the kernel $H_\lambda^{k,l}$ for $k \geqq l + (l-1)/\varepsilon m$ verifies the estimate:

$$(6.22)\qquad |H_\lambda^{k,l}(x,y)| \leqq C\frac{|\lambda|^{n/m}}{d(\lambda)}\cdot\left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l}$$

for $d(\lambda) \geqq \max\{|\lambda|^{1-\theta/m}, |\lambda|^{1-1/m+\varepsilon}\}$, $|\lambda| \geqq 1$, and x satisfying (6.15).

Summing up we have proved the following result.

THEOREM 6.1. *The kernel $G_\lambda(x,y)$ of the resolvent $R_\lambda = (\tilde{A} - \lambda)^{-1}$ (considered as an operator in $L_2(R^n)$) has an asymptotic representation of the form*:

$$(6.23)\qquad \begin{aligned} G_\lambda(x,y) = F_\lambda(x,y) &+ \sum_{r=1}^{l}\ \sum_{J \in S(r,k)} ([A_0 - A, A_0; J]F^{|J|+r+1})(x,y) \\ &+ O\left(\frac{|\lambda|^{n/m}}{d(\lambda)}\cdot\left(\frac{|\lambda|^{1-\theta/m}}{d(\lambda)}\right)^{l+1}\right), \end{aligned}$$

with $A_0 = A'(x^0, D)$ (x^0 a fixed point), $F_\lambda = (\tilde{A}_0 - \lambda)^{-1}$, such that:

(i) *If A' has constant coefficients then $\theta = 1$, k and l any integers with $k \geqq l - 1 \geqq 0$; the O estimate holds for $\lambda \to \infty$ in the region*

$$d(\lambda) \geqq |\lambda|^{(m-1)/m}, \quad |\lambda| \geqq 1,$$

uniformly in x, y and x^0.

(ii) *If A' has variable coefficients then θ is given by* (6.12), *k and l any positive integers with $k/l \geqq 1 + (\varepsilon m)^{-1}$ for any given $\varepsilon > 0$; the O estimate holds for $\lambda \to \infty$ in the region*: $d(\lambda) \geqq \max\{|\lambda|^{1-(\theta/m)}, |\lambda|^{1+\varepsilon-(1/m)}\}$, $|\lambda| \geqq 1$, *for any y*

but x restricted to the neighborhood (6.15) *of x^0. Under these restrictions the O estimate is uniform in x, y and x^0.*

The asymptotic representation formula of $G_\lambda(x,y)$ takes a particularly simple form on the diagonal of $R^n \times R^n$. As a matter of fact in this case (6.23) can be replaced by an asymptotic series expansion in powers of $(-\lambda)^{-1/m}$. To see this take in (6.23) $x = y = x^0$. From (4.7) it follows that

$$(-\lambda)^{1-n/m}([A_0 - A, A_0; J]F_\lambda^{|J|+r+1})\ (x_0, x_0) \tag{6.24}$$

is a polynomial in $(-\lambda)^{-1/m}$ with coefficients which are C_*^∞ functions in x^0. It is easy to check (using Theorem 5.2) that the constant term in this polynomial is zero. Also, $F_\lambda(x^0,x^0) = g_0(x^0)\,(-\lambda)^{-1+(n/m)}$ with

$$g_0(x) = (2\pi)^{-n}\int_{R^n}[A'(x,\xi)+1]^{-1}d\xi. \tag{6.25}$$

These observations and Theorem 6.1 show that on the diagonal the kernel G_λ has the asymptotic series expansion:

$$G_\lambda(x^0,x^0) \sim (-\lambda)^{n/m-1}\sum_{j=0}^{\infty} g_j(x^0)\,(-\lambda)^{-j/m} \tag{6.26}$$

with coefficients g_j which are C_*^∞ functions in x^0 (g_j for $j > 0$ is the sum of the coefficients of $(-\lambda)^{-j/m}$ in the polynomials (6.24) taken over all $J \in S(r,k)$, $r = 1,\cdots,l$ where l is chosen large enough so that $(l+1)\theta > mj$). The asymptotic expansion (6.26) holds in the usual sense for $\lambda \to \infty$ in the region:

$$d(\lambda) \geqq |\lambda|^{1-(1/m)+\varepsilon}, \qquad |\lambda| \geqq 1,$$

if A' has constant coefficients, and for $\lambda \to \infty$ in the region

$$d(\lambda) \geqq |\lambda|^{1-(\theta/m)+\varepsilon}, \qquad |\lambda| \geqq 1,$$

in the general case. Here ε is an arbitrary fixed positive number. The asymptotic expansion in the regions mentioned is uniform in $x^0 \in R^n$.

Recall that the standard resolvent kernel of $\tilde{A}$ is the kernel

$$R_\lambda(x,y) = \rho(y)^{-1}G_\lambda(x,y)$$

(the kernel with respect to the measure $d_\rho y = \rho(y)dy$). The asymptotic expansion which we have derived for $R_\lambda(x,x)$ (via (6.26)) is precisely the asymptotic expansion (3.2) of Theorem 3.1. Thus we have proved Theorem 3.1 for the class of operators $\tilde{A}$ in $L_{2,\rho}(R^n)$.

We now extend Theorem 6.1 to the case of a self-adjoint realization of an elliptic differential operator on Ω for any open set $\Omega \subset R^n$.

THEOREM 6.2. *Let $\tilde{A}$ be a positive self-adjoint operator in $L_{2,\rho}(\Omega)$ which is the realization of a ρ-formally self-adjoint (positive) elliptic operator $A(x,D)$*

of order $m > n$ (ρ and the coefficients of A belong to $C^\infty(\Omega)$). Let $R_\lambda(x,y)$ be the resolvent kernel of $\tilde{A}$. Then the conclusion of Theorem 6.1 holds for the kernel $G_\lambda(x,y) = \rho(y)R_\lambda(x,y)$ with the modification that the statements on the uniform dependence of the O estimate in (6.23) hold for x, y and x^0 restricted to any compact subset of Ω.

As above Theorem 6.2 yields for $G_\lambda(x^0,x^0)$ the asymptotic series expansion (6.26) valid for $\lambda \to \infty$ in the region: $d(\lambda) \geqq |\lambda|^{1-(1/m)+\varepsilon}$, $|\lambda| \geqq 1$, if A' has constant coefficients, and in the region: $d(\lambda) \geqq |\lambda|^{1-(\theta/m)+\varepsilon}$, $|\lambda| \geqq 1$, in the general case (the expansion being uniform in x^0 in every compact subset of Ω). Now, the existence of such an expansion is precisely the statement of Theorem 3.1. Thus we see that Theorem 6.2 implies Theorem 3.1 as a special case.

For the proof of Theorem 6.2 we shall need the following:

LEMMA 6.1. *For every complex λ which is not on the non-negative axis, let T_λ be a bounded linear operator in $L_2(\Omega)$ such that its range and the range of its adjoint T_λ^* are contained in $H_m^{loc.}(\Omega)$, $m > n$. By Theorem 2.1 bis. T_λ is an integral operator with a continuous kernel $K_\lambda(x,y)$. Suppose that*

$$\| T_\lambda \| \leqq \frac{c}{d(\lambda)}, \qquad c \text{ a constant.} \tag{6.27}$$

Suppose moreover that there exist positive elliptic differential operators $A(x,D)$ and $A_1(x,D)$ (C^∞ coefficients) of order m such that

$$(A(x,D) - \lambda)T_\lambda f = 0 \text{ and } (A_1(x,D) - \lambda)T_\lambda^* f = 0 \tag{6.28}$$

for all $f \in L_2(\Omega)$, Then for every integer $j \geqq 0$ and every $\Omega_0 \subset\subset \Omega$ (i.e. Ω_0 open, $\bar{\Omega}_0$ compact and $\bar{\Omega}_0 \subset \Omega$), the following estimate holds:

$$\sup_{\Omega_0 \times \Omega_0} |K_\lambda(x,y)| \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^j, \qquad |\lambda| \geqq 1, \tag{6.29}$$

where C is a constant independent of λ.

Proof. It will suffice to prove (6.29) for Ω_0 with a smooth boundary. We first prove that for $\Omega_0 \subset\subset \Omega$ and $j = 0, 1, 2, \cdots$:

$$\begin{aligned} \| T_\lambda f \|_{0,\Omega_0} &\leqq \frac{C}{d(\lambda)} \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^j \| f \|_{0,\Omega}, \\ \| T_\lambda f \|_{m,\Omega_0} &\leqq C \frac{|\lambda|}{d(\lambda)} \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)}\right)^j \| f \|_{0,\Omega} \end{aligned} \tag{6.30}$$

for all $f \in L_2(\Omega)$. Here and in the following $C, C_1, \cdots$, denote constants independent of λ or f. Indeed, write $u = T_\lambda f$ and note that by assumption: $Au = \lambda u$. Using well known a-priori estimates for solutions of elliptic equations we have:

$$\begin{aligned}(6.31)\qquad \| u \|_{m,\Omega_0} &\leqq C_1(\| Au \|_{0,\Omega} + \| u \|_{0,\Omega}) \\ &= C_1(|\lambda| + 1) \| u \|_{0,\Omega} \leqq C \frac{|\lambda|}{d(\lambda)} \| f \|_{0,\Omega}\end{aligned}$$

for $|\lambda| \geqq 1$, by (6.27). (6.31) together with (6.27) yield (6.30) for $j = 0$. We continue by induction. Suppose that (6.30) was proved for j we shall prove it for $j+1$, To this end observe that our assumption that A is a positively elliptic operator implies that A can be written in the form: $A = A^0 + B$ where B is an operator of order $\leqq m-1$ and A^0 is a formally self-adjoint operator such that

$$(6.32)\qquad \| v \|_{0,\Omega} \leqq d(\lambda)^{-1} \| (A^0 - \lambda) v \|_{0,\Omega}$$

for all $v \in H_m(\Omega)$ with compact support in Ω. Now, given Ω_0 choose Ω_1 with a smooth boundary such that $\Omega_0' \subset\subset \Omega_1 \subset\subset \Omega$. Pick $\zeta \in C_0^\infty(\Omega_1)$ such that $\zeta \equiv 1$ on Ω_0. Write as before $u = T_\lambda f$ and apply (6.32) to $v = \zeta u$. We have:

$$\begin{aligned}(6.33)\qquad \| u \|_{0,\Omega_0} &\leqq \| \zeta u \|_{0,\Omega} \leqq d(\lambda)^{-1} \| (A^0 - \lambda)(\zeta u) \|_{0,\Omega} \\ &\leqq d(\lambda)^{-1} \| (A - \lambda)(\zeta u) \|_{0,\Omega} + C_2 d(\lambda)^{-1} \| u \|_{m-1,\Omega_1} \\ &\leqq C_3 d(\lambda)^{-1} \| u \|_{m-1,\Omega_1},\end{aligned}$$

since $(A - \lambda)u = 0$. We now apply the well known interpolation inequality:

$$(6.34)\qquad \| u \|_{m-1,\Omega_1} \leqq \gamma \| u \|_{0,\Omega_1}^{1/m} \| u \|_{m,\Omega_1}^{1-1/m},$$

γ a constant. Using our induction assumption, it follows from (6.30) (applied to Ω_1) and (6.34) that

$$(6.35)\qquad \| u \|_{m-1,\Omega_1} \leqq \gamma C \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)} \right)^{j+1} \| f \|_{0,\Omega}.$$

Combining (6.33) and (6.35) we obtain the first inequality (6.30) for $j+1$. To derive the second inequality we use again the interior a-priori estmates:

$$\begin{aligned}(6.36)\qquad \| u \|_{m,\Omega_0} &\leqq C_3(\| Au \|_{0,\Omega_1} + \| u \|_{0,\Omega_1}) \\ &= C_3(|\lambda| + 1) \| u \|_{0,\Omega_1} \leqq 2C_3 |\lambda| \| T_\lambda f \|_{0,\Omega_1}, \quad |\lambda| \geqq 1.\end{aligned}$$

Combining (6.36) with the first inequality (6.30) with j replaced by $j+1$ (which we have just established for *all* $\Omega_0 \subset\subset \Omega$) we obtain the second inequality (6.30) for $j+1$. This completes the proof of (6.30).

Let, now, $J_0 : L_2(\Omega) \to L_2(\Omega_0)$ be the restriction operator from Ω to Ω_0 (Ω_0 as above with a smooth boundary). Its adjoint $J_0^* : L_2(\Omega_0) \to L_2(\Omega)$ is an extension operator. We define: $T_{\lambda,0} = J_0 T_\lambda J_0^*$. It is clear that $T_{\lambda,0}$ is a bounded operator in $L_2(\Omega_0)$ which verifies the conditions of Theorem 2.1 (its kernel is $K_\lambda(x,y)$ restricted to $\Omega_0 \times \Omega_0$). From (6.30) it follows that

$$\text{(6.37)} \qquad \| T_{\lambda,0} \|_{0,\Omega_0} \leqq \frac{C}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)} \right)^j,$$

$$\| T_{\lambda,0} \|_{m,\Omega_0} \leqq C \frac{|\lambda|}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)} \right)^j.$$

Since T_λ^* is an operator with the same properties as T_λ, the estimates (6.30) also hold for T_λ^*. Hence:

$$\text{(6.37)}' \qquad \| T_{\lambda,0}^* \|_{m,\Omega_0} \leqq C \frac{|\lambda|}{d(\lambda)} \cdot \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)} \right)^j.$$

Applying now the inequality (2.2) to the kernel of $T_{\lambda,0}$, using the norm estimates (6.37) and (6.37)′ we obtain (6.29). This establishes the lemma.

We conclude with the

Proof of Theorem 6.2. Let $\Omega_0 \subset\subset \Omega$. Choose a real function $\rho^0(x) \in C_*^\infty(R^n)$ such that $\rho^0(x) \equiv \rho(x)$ on Ω_0, $\rho^0(x) \geqq \delta > 0$ on R^n, and then choose a ρ^0-formally self-adjoint uniformly elliptic operator $A^0(x, D)$ on R^n with C_*^∞ coefficients such that A^0 coincides with the given elliptic operator A on Ω_0. (The proof that the extension A^0 of A exists is standard). Let $\tilde{A}^0$ be the unique self-adjoint realization of A^0 in $L_{2,\rho^0}(R^n)$. $\tilde{A}^0$ is semi-bounded from below. Without loss of generality we shall assume that $\tilde{A}^0$ is a positive operator as this may always be achieved by adding a large positive multiple of the identity to both operators A^0 and A. Let $R_\lambda^0(x, y)$ be the resolvent kernel of $\tilde{A}^0$ and let $G_\lambda^0(x, y) = \rho^0(y) R_\lambda^0(x, y)$. We define operators S_λ and S_λ^0 from $L_2(\Omega_0)$ into $L_2(\Omega_0)$ by:

$$S_\lambda f = \int_{\Omega_0} R_\lambda(x, y) f(y) dy, \quad S_\lambda^0 f = \int_{\Omega_0} R_\lambda^0(x, y) f(y) dy,$$

$f \in L_2(\Omega_0)$. We set:

$$T_\lambda = S_\lambda - S_\lambda^0.$$

It is easily seen that T_λ verifies the conditions of Lemma 6.1. Indeed, T_λ is a bounded linear operator in $L_2(\Omega_0)$ with range in $H_m(\Omega_0)$. The same is true for its adjoint since $T_\lambda^* = T_{\bar{\lambda}}$. That the estimate (6.27) holds is obvious from the relation of S_λ and S_λ^0 to the resolvent operators of $\tilde{A}$ and $\tilde{A}_0$. For $f \in L_2(\Omega_0)$ we have:

$$(A - \lambda) T_\lambda f = (A - \lambda) S_\lambda f - (A^0 - \lambda) S_\lambda^0 f = \frac{f}{\rho} - \frac{f}{\rho^0} = 0,$$

since $A = A^0$ and $\rho = \rho^0$ on Ω_0. Similarly, $(A - \bar{\lambda}) T_\lambda^* f = 0$. Hence, applying Lemma 6.1 to the kernel of T_λ we find that for every $\Omega_1 \subset\subset \Omega_0$ and every integer $j \geqq 0$, the following estimate holds:

$$\sup_{\Omega_1 \times \Omega_1} |G_\lambda(x,y) - G_\lambda^0(x,y)| \leqq C \frac{|\lambda|^{n/m}}{d(\lambda)} \left(\frac{|\lambda|^{1-1/m}}{d(\lambda)} \right)^j, \quad |\lambda| \geqq 1. \tag{6.38}$$

By Theorem 6.1 the kernel $G_\lambda^0(x,{}_{,}y)$ has the asymptotic representation (6.23). Combining this with (6.38) (taking $j = l + 1$), it follows that the asymptotic formula (6.23) also holds for the kernel $G_\lambda(x\ y)$. This proves the theorem.

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